Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 6, pp. 1101–1110, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)00142-1

ON THE CLASSICAL THEORY OF PLATES†

K. Yu. VOLOKH

Moscow

(*Received* 29 *July* 1993)

In the traditional description of the classical theory of plates the shear forces, by definition, are integrals over the plate thickness of the transversal shear stresses. This leads to a breakdown of Hooke's law for these stresses, inconsistency of the equations of equilibrium of the classical theory of plates with the principle of virtual work and the occurrence of contradictions of the type which exist in the problem of a beam bent statically by equivalent loads. These drawbacks of the traditional description can be eliminated if we assume that the shear forces, by definition, are statically equivalent to "rotated" bending and twisting moments (which, in the case of the classical theory of plates, is not related to the St Venant principle). This treatment of the shear forces is the basis of a proposed version of the description of the classical theory of plates. An analysis is also given of publications in which doubts are expressed regarding the correctness of the classical theory of plates. It is shown that the arguments put forward in these publications do not justify dispensing with the classical theory of plates.

1. THE PRINCIPLE OF VIRTUAL WORK AND THE EQUATIONS OF EQUILIBRIUM

For a continuous deformed body, occupying a volume V and bounded by a surface 0, for small deformations in rectangular Cartesian coordinates, x, y, z, the principle of virtual work can be written in the form

$$\delta' U = \delta' U_1 + \delta' U_2 = 0 \tag{1.1}$$

where

$$\delta' U_1 = \iiint_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz})dV \tag{1.2}$$

$$\delta' U_2 = -\iiint_V (F_x^* \delta u + F_y^* \delta v + F_z^* \delta w)dV - \iint_0 (P_x^* \delta u + P_y^* \delta v + P_z^* \delta w)dO \tag{1.3}$$

$$\varepsilon_x = \partial u / \partial x,\ \ \varepsilon_y = \partial v / \partial y,\ \ \varepsilon_z = \partial w / \partial z$$
$$\gamma_{xy} = \partial u / \partial y + \partial v / \partial x,\ \ \gamma_{xz} = \partial w / \partial x + \partial u / \partial z,\ \ \gamma_{yz} = \partial w / \partial y + \partial v / \partial z \tag{1.4}$$

Here F_x^*, F_x^*, F_x^* and P_x^*, P_x^*, P_x^* are the volume and surface forces. The asterisk denotes that these quantities are specified. We will use the symbol δ' to denote the virtual work so as to emphasize that we are not concerned with a variation of the functional.

The principle of virtual work (1.1) is a fundamental variational principle (not in the sense of a variation of a functional but in the sense of work on variations of the displacements), and the equations of equilibrium are obtained from it by identify transformations without introducing any relationships between the strains and stresses.

We will obtain the equations of equilibrium for a body in the form of a plate of thickness h. To do this we first convert (1.2) and (1.3) in accordance with the kinematic assumptions made in the classical theory of plates

$$\varepsilon_z = 0,\ \ \gamma_{xz} = 0,\ \ \gamma_{yz} = 0$$

With these assumptions we obtain from (1.4)

†*Prikl. Mat. Mekh.* Vol. 58, No. 6, pp. 156–165, 1994.

$$w = w(x,y),\ u = \vartheta_x(x,y)z,\ \upsilon = \vartheta_y(x,y)z \tag{1.5}$$

where $\vartheta_x = -\partial w/\partial x$, $\vartheta_y = -\partial w/\partial y$ are the angles of rotation of the normals to the median plane.

We used the conditions $u(z = 0) = 0$, $\upsilon(z = 0) = 0$ in deriving (1.5). This, however, is not an additional assumption, since it can be shown that the equations of equilibrium in the classical theory of plates can be separated into a plane problem and a bending problem. We will only consider the latter below.

By the assumptions made above, Eqs (1.2) and (1.3) can be converted to the form

$$\delta' U_1 = \iiint_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy})dV \tag{1.6}$$

$$\delta' U_2 = -\iint_\Omega (q\delta w + m_x \delta\vartheta_x + m_y \delta\vartheta_y)d\Omega - \int_S (Q^*\delta w - M_\nu^* \delta w_{,\nu} - M_s^* \delta w_{,s})dS \tag{1.7}$$

where

$$q = \int F_z^* dz + \sigma_z^*(h/2) - \sigma_z^*(-h/2)$$

$$m_x = h[\tau_{xz}^*(h/2) - \tau_{xz}^*(-h/2)],\ m_y = h[\tau_{yz}^*(h/2) - \tau_{yz}^*(-h/2)]$$

$$Q^* = \int \tau_{\nu z}^* dz,\ M_\nu^* = \int \sigma_\nu^* z dz,\ M_s^* = \int \tau_{s\nu}^* z dz$$

Here Ω is the area of the median plane of the plate, bounded by the contour S. The normal to the contour ν is specified by the direction cosines $\cos(x, \nu) = \cos\varphi$, $\cos(y, \nu) = \sin\varphi$. Since, according to the kinematic relations imposed $\delta\varepsilon_z = \delta\gamma_{xz} = \delta\gamma_{yz} = \delta u(z = 0) = \delta\upsilon(z = 0) = 0$, the corresponding terms in (1.6) and (1.7) are omitted. It should be noted immediately that terms containing τ_{xz}, τ_{yz}, σ_z play no further part in our subsequent analysis, and these stresses will therefore not occur in the equations of equilibrium.

We convert (1.6) to the following form

$$\delta' U_1 = \iiint_V \left\{ \sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta \upsilon}{\partial y} + \tau_{xy} \frac{\partial \delta u}{\partial y} + \tau_{xy} \frac{\partial \delta \upsilon}{\partial x} \right\} dV =$$

$$= \iiint_V \left\{ \frac{\partial}{\partial x}(\sigma_x \delta u) + \frac{\partial}{\partial y}(\sigma_y \delta \upsilon) + \frac{\partial}{\partial y}(\tau_{xy}\delta u) + \frac{\partial}{\partial x}(\tau_{xy}\delta\upsilon) \right\} dV -$$

$$-\iiint_V \left\{ \frac{\partial \sigma_x}{\partial x}\delta u + \frac{\partial \sigma_y}{\partial y}\delta\upsilon + \frac{\partial \tau_{xy}}{\partial y}\delta u + \frac{\partial \tau_{xy}}{\partial x}\delta\upsilon \right\} dV \tag{1.8}$$

W substitute (1.5) here, and we similarly get rid of the variation under the derivative sign. We further integrate over the thickness and, using Gauss' formula, we obtain the following final expression for the virtual work

$$\delta' U = -\iint_\Omega \left\{ \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} + q \right\} \delta w d\Omega -$$

$$-\int_S \left\{ (M_\nu - M_\nu^*)\delta w_{,\nu} + (M_s - M_s^*)\delta w_{,s} - (Q - Q^* + m)\delta m \right\} ds = 0 \tag{1.9}$$

$$M_x = \int \sigma_x z dz,\ M_y = \int \sigma_y z dz,\ M_{xy} = \int \tau_{xy} z dz,\ m = m_x \cos\varphi + m_y \sin\varphi$$

$$M_\nu = M_x \cos^2\varphi + M_y \sin^2\varphi + M_{xy}\sin 2\varphi,\ M_s = \frac{1}{2}(M_y - M_x)\sin 2\varphi + M_{xy}\cos 2\varphi$$

$$Q = \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right)\cos\varphi + \left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \right)\sin\varphi$$

Since (1.9) is the equation of the work of the external and internal forces for virtual displacements, the factor in front of δw in the first integral is the projection onto the z axis of the external and internal forces acting on an element of the plate $hd\Omega$. The condition of equilibrium of this element has the form

$$\frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}\right)=-q-\frac{\partial m_x}{\partial x}-\frac{\partial m_y}{\partial y} \tag{1.10}$$

We will now consider the contour integral in (1.9). We denote the integrand by L. Since we completely define $\delta w_{,s}$ by specifying δw on the contour, the third term must be converted by integrating by parts

$$-\int L ds=-\int \Phi ds-(M_s-M_s^*)\delta w\Big|_{C_1^-}^{C_1^+}$$
$$\Phi=(M_\nu-M_\nu^*)\delta w_{,\nu}-\left(Q+\frac{\partial M_s}{\partial s}-Q^*+m-\frac{\partial M_s^*}{\partial s}\right)\delta w \tag{1.11}$$

The term outside the integral is evaluated as the difference between the values of the corresponding functions at the ends of the contour C_1, where the force conditions are specified.

We will now consider the case of the corner point s^* on the contour, when φ changes abruptly

$$-\int_{C_1} L ds=\lim_{a\to 0}\left\{-\int_{C_1^-}^{s^*-a} L ds-\int_{s^*-a}^{s^*+a} L ds-\int_{s^*+a}^{C_1^+} L ds\right\}-Q^*(s^*)\delta w(s^*) \tag{1.12}$$

The occurrence of the last term is due to the possibility of a concentrated action being present at the corner point, which is taken into account in L by means of a delta function. We finally obtain

$$-\int_{C_1} L ds=-\int_{C_1^-}^{s^*-0}\Phi ds-\int_{s^*+0}^{C_1^+}\Phi ds-(M_s-M_s^*)\delta w\Big|_{C_1^-}^{C_1^+}+$$
$$+\left[M_s(s^*+0)-M_s(s^*-0)+M_s^*(s^*-0)-M_s^*(s^*+0)-Q^*(s^*)\right]\partial w(s^*) \tag{1.13}$$

Equating the expression in square brackets to zero, we obtain the conditions at the discontinuity. Similar conditions, derived in [2, 5], do not contain the term $Q^*(s^*)$ due to the fact that the possibility of the application of concentrated forces was not taken into account.

Hence, the following conditions are specified on the smooth part of the contour

$$w=w^*\leftrightarrow Q+\frac{\partial M_s}{\partial s}=Q^*+\frac{\partial M_s^*}{\partial s}-m,\quad w_{,\nu}=w_{,\nu}^*\leftrightarrow M_\nu=M_\nu^* \tag{1.14}$$

and at the point of inflection of the contour we also have the following condition at the discontinuity

$$w(s^*)=w^*(s^*)\leftrightarrow M_s(s^*+0)-M_s(s^*-0)=M_s^*(s^*+0)-M_s^*(s^*-0)+Q^*(s^*) \tag{1.15}$$

Since the terms occurring in (1.13) have a clear physical meaning, namely, they represent the work done in deforming the contour of the plate, the factors in front of the virtual generalized displacements are also corresponding generalized contour forces. Hence, an external shear contour force $K^*=Q^*+\partial M_s/\partial s$ and a moment M_ν^* act on the contour, to which there correspond internal force factors $K=Q+\partial M_s/\partial s$ and M_ν. For the corner point, the shear contour forces $K^\vee$ and $K^{\vee *}$ are defined, respectively, by the left- and right-hand sides of (1.15). As can be seen, the same contour force K^* can be specified differently in terms of Q^* and M_s^*.

The equations of the equilibrium of the internal elements of the plate (1.10) and the contour elements of the plate (1.14) and (1.15) obtained above hold for any relations between the stresses and strains, including the case when the stresses cannot be expressed directly as a function of the strains, for example, in the theory of plastic flow, etc.

In the case of an isotropically elastic body we have

$$\sigma_x=E(\varepsilon_x+\mu\varepsilon_y)/(1-\mu^2),\quad \sigma_y=E(\varepsilon_y+\mu\varepsilon_x)/(1-\mu^2),\quad \tau_{xy}=E\gamma_{xy}/2(1+\mu) \tag{1.16}$$

In Hooke's law it is assumed that $\sigma_z = 0$. Substituting (1.16) into (1.10), (1.14) and (1.15) and using (1.4) and (1.5) we obtain the Germain–Lagrange equation and the force contour conditions in displacements. Note that we are not concerned here with any breakdown of Hooke's law for τ_{xz} and τ_{yz}.

We now need to look at two aspects of the proposed derivation of the boundary-value problem of the classical theory of plates.

First, the transverse shear stresses τ_{xz} and τ_{yz} do not participate in the balance of forces acting on an element of the plate since they do not do work on the zero deformations γ_{xz} and γ_{yz}. Kirchhoff, Boussinesq, Clebsch, St Venant, Kelvin and Tait [1] also assumed $\tau_{xz} = \tau_{yz} = 0$, although this is not related to the principle of virtual work. However, they did not attempt to explain how, in this case, the balance of the external shear loads occurs, which, obviously, also led to the modern (traditional) description of the classical theory of plates, when integrals over the thickness of the transverse shear stresses $Q_x \overset{\text{def}}{=} \int \tau_{xz} dz (x \rightleftarrows y)$ are considered as the internal shear forces which balance the external load. This treatment of the shear forces leads to deficiencies in the description, indicated in the abstract above. An attempt to establish a mechanism by which the external shear loads are balanced without giving up the idea that there are no shear stresses τ_{xz} and τ_{yz}, is made in the next section.

Second there is only one equation of equilibrium (1.10) for a non-contour element of the plate—there are no equations of the balance of the moments in the projection on the z axis relative to the x and y axes. This does not mean that balance of the moments breaks down. A detailed discussion is given below. We merely note that this situation is not unusual. Thus, the equations of the classical theory of elasticity in displacements consists of three equations of equilibrium in projections onto the x, y and z axes. Although the equations of the balance of the moments about these axes does not occur in explicit form, the balance of the moments is taken into account automatically via the "pairing law" of the shear stresses.

2. INTERPRETATION OF THE SHEAR TRANSVERSE FORCES

According to the above assumptions, within the framework of the classical theory of plates a plate can be regarded as a combination of an infinitely large number of absolutely rigid and infinitely small, in plan, prisms of height h.

By saying that the prisms are absolutely rigid we have in mind the possibility of statistically equivalent transformations, described later, but this does not mean that there are strains ε_x, ε_y, γ_{xy} in planes parallel to the median plane of the plate and that the conditions of continuity of the strains break down.

Bending moments M_x and M_y, torques M_{xy}, distributed external moments m_x and m_y, and a distributed external load q (Fig. 1) act on the prism. Summing the moments, which rotate the prism around the x and y axes, and neglecting higher-order terms, we obtain the following expression for the resultant moments

$$\begin{aligned} G_x dxdy &\overset{\text{def}}{=} \frac{\partial M_x}{\partial x} dxdy + \frac{\partial M_{xy}}{\partial y} dxdy + m_x dxdy \\ G_y dxdy &\overset{\text{def}}{=} \frac{\partial M_y}{\partial x} dxdy + \frac{\partial M_{xy}}{\partial y} dxdy + m_y dxdy \end{aligned} \tag{2.1}$$

Since each prism is an absolutely "rigid body", a statically equivalent transformation of the moments, illustrated in Fig. 2, is legitimate. For transformations in the x direction we must assume $H \overset{\text{def}}{=} G_x$, $l \overset{\text{def}}{=} x$, while in the y direction we must assume $H \overset{\text{def}}{=} G_y$, $l \overset{\text{def}}{=} y$. Hence, the load which is balanced by the distributed forces $\partial G_x/\partial x$ and $\partial G_y/\partial_y$ in the internal region of the plate

$$\partial G_x / \partial x + \partial G_y / \partial y + q = 0 \tag{2.2}$$

or, separating the average internal moment $Q_x dxdy$, Q_{ydxdy} from the external moment $m_x dxdy$ and $m_y dxdy$, we obtain the equation of equilibrium of the element of the plate on which an external shear load q^* acts, which is balanced by the internal shear forces Q_x and Q_y in the projection on the z axis (Fig. 3)

$$\partial Q_x / \partial x + \partial Q_y / \partial y + q^* = 0 \tag{2.3}$$

Here

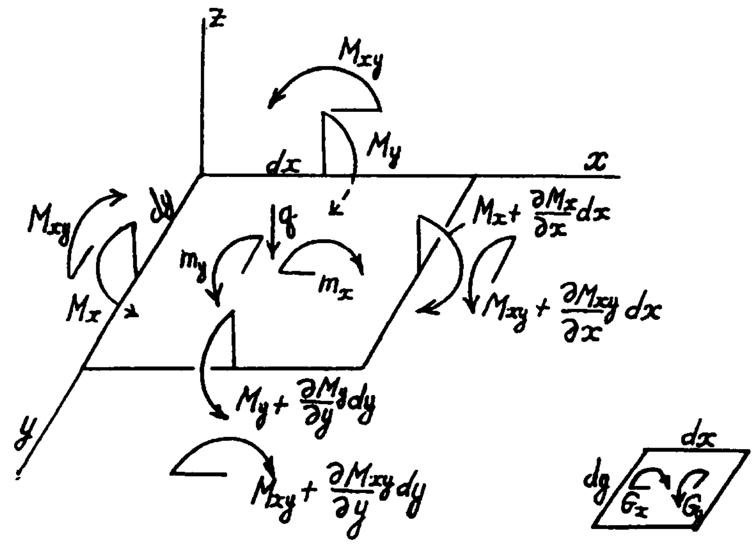

Fig. 1.

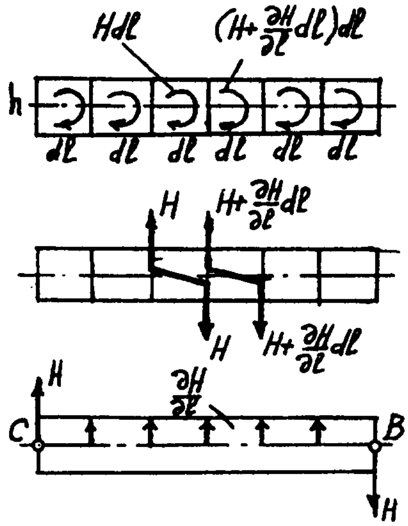

Fig. 2.

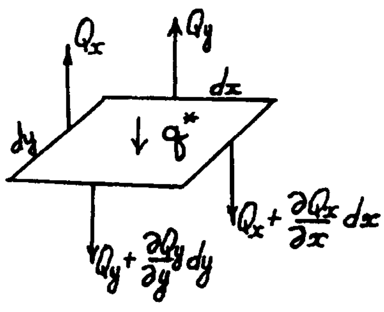

Fig. 3.

$$Q_x \overset{\text{def}}{=} \partial M_x / \partial x + \partial M_{xy} / \partial y, \; Q_y \overset{\text{def}}{=} \partial M_y / \partial y + \partial M_{xy} / \partial x \tag{2.4}$$

$$q^* \overset{\text{def}}{=} q + \partial m_x / \partial x + \partial m_y / \partial y$$

are definitions of the transverse forces as "rotated" moments. Relations (2.4) are not the equations of the balance of moments, they are equalities by definition. After all the moments, internal and external, acting on the element of the plate are "rotated", the need to consider the balance conditions separately drops out.

Equation (2.3) is the necessary and sufficient condition for self-balancing of an element of the plate. The self-balancing condition without using conversion of the moments would have the form

$$G_x = 0, \; G_y = 0, \; q = 0 \tag{2.5}$$

which is of no practical interest although it is also sufficient for self-balancing of the elements. Note that similar discussions on the conditions for self-balancing of a chain of rigid cross-shaped elements were used in [6, pp. 69–71] to illustrate the Kelvin–Tait transformation, about which we shall say more below.

We will now consider the conditions on the contour s. We obtain from the equilibrium of the contour elements (Fig. 4)

$$M_\nu = M_\nu^*, \; Q = Q^* - m, \; M_s = M_s^* \tag{2.6}$$

where M_ν, Q and M_s are given by (1.9). Further using the statically equivalent transformation for $M_s(H \overset{\text{def}}{=} M_s$, $l \overset{\text{def}}{=} s)$, proposed for the first time by Kelvin and Tait, we obtain the boundary conditions (1.14) and (1.15).

It should be borne in mind that not only moments but also transverse forces $G_x = Q_x + m_x$ and $G_y = Q_y + m_x$ are applied over the surfaces of the contour elements. This is a consequence of the fact that these surfaces are finite points for the statically equivalent transformations in the inner regions, i.e. the points B or C in Fig. 2.

We emphasize the differences between the proposed treatment of the classical theory of plates and the traditional one. First, the "rotated" moments (2.4) are the shear forces in the proposed treatment, whereas in the traditional treatment the shear forces are the integrals of the shear stresses

$$Q_x \overset{\text{def}}{=} \int \tau_{xz} dz, \; Q_y \overset{\text{def}}{=} \int \tau_{yz} dz \tag{2.7}$$

Second, in the traditional treatment, Eqs (2.4), after changing the sign of equality by definition ($\overset{\text{def}}{=}$) into the sign of simple equality, are the equations of the balance of the moments. Hence, in the traditional treatment of the classical theory of plates we have the equations of equilibrium. In the treatment of the classical theory of plates proposed above there is no need to consider the equations of the balance of the moments separately.

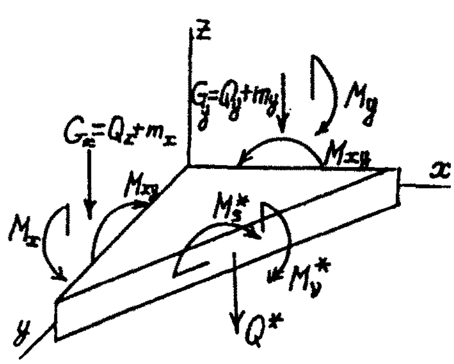

Fig. 4.

It can be seen from the above discussion that the treatment of the classical theory of plates proposed in this section agrees well with the principle of virtual work. Note also that the hypotheses of undeformed normals used in the classical theory of plates leads to the need for statically equivalent transformations. These transformations are in no way related to the St Venant principle and due solely to the kinematic assumptions made. The problem of the statically equivalent transformations on the contour was solved a long time ago; the most exhaustive discussion of this can be found in [6, pp. 58–71]. The method of constructing the classical theory of plates proposed above consists essentially of extending the Kelvin–Tait transformation to the internal region of the plate.

3. SOME PROBLEMS IN THE CLASSICAL THEORY OF PLATES

We will now consider some examples [1–3] which were the reason for the assertion that the classical theory of plates "is unable to obtain a correct solution of certain problems which, in their formulation, are outside the scope of the hypotheses employed", and, consequently, "raise natural doubts as to its completeness as a physical theory" [1].

The problem of the torsion of a plate. "In general, there is no solution within the framework of the classical theory of plates of the problem of the free torsion of a rectangular plate, the transverse edges of which are loaded with torques, while the longitudinal edges are free. As we know, by the St Venant solution there are no normal stresses in a plate loaded in this way, and, consequently, no bending moments or curvatures. As a result, its deflection takes the form $w = cxy$ (where c is a constant), corresponding to torsional moments acting both on the transverse and longitudinal edges. It turns out to be impossible to ensure that there are no such moments on the longitudinal edges, i.e. to satisfy the specified boundary conditions in the classical theory of plates" [1].

Substituting $w = cxy$ into the contour conditions (1.14) and (1.15) and taking Hooke's law into account, we obtain

$$K^* = Q^* + \frac{\partial M_s^*}{\partial s} - m = 0, \quad M_\nu^* = 0 \tag{3.1}$$

For the corner point

$$K^{\nu *} = M_s^*(s^* + 0) - M_s^*(s^* - 0) + Q^*(s^*) = -2(1-\mu)cD, \quad D = \frac{Eh^3}{12(1-\mu^2)} \tag{3.2}$$

Equations (3.1) and (3.2) uniquely define the external contour forces of the load, to which corresponds the specified value of the deflection. In other words, the plate bends along the surface $w = cxy$ due to the application of concentrated pairwise opposite forces at the corners of the plate. It can be seen from (3.2) that the specified quantity $K^{\nu *}$ can be expressed differently in terms of M_s^* and Q^*. Thus we can assume that the same torsional moments M_s^* act on the plate on all sides, while the external torsional moments M_s^* act only along opposite sides, etc. All these versions of the application of external loads are statically equivalent. Hence, the idea of a "torque" on the contour is meaningless in the classical theory of plates: it is impossible to distinguish it from the Kirchhoff shear force on the contour. We cannot say whether there is or is not a torque on the contour; we can only say that Kirchhoff forces are or are not present. The various problems of the torsion of a plate in the formation of the three-dimensional theory of elasticity under pairwise opposite corner forces or torques from opposite sides or other problems for which conditions (3.1) and (3.2) hold, "merge" into one in the classical theory of plates. This illustrates the approximate nature of the classical theory of plates but not its internal contradiction.

Regarding the order of the equations of the classical theory of plates. "... in mechanics the order of the equation n and the number of boundary conditions m are closely related to the number of conservation laws k employed, namely, $n = 2k$ and $m = k$. In the theory of the bending of plates three laws of conservation are used: the balance of the shear forces and two equations of the balance of the moments. This means that the theory of plates should be described by sixth-order equations with three boundary conditions". In the classical theory of plates "which is described by fourth-order equations with two boundary conditions, one of the conservation laws is lost: the equation of the balance of the shear forces, strictly speaking, is not satisfied" [2].

The applicability of the hypothesis regarding the relation between the order of the equations and the number of conservation laws is obviously confined to higher powers than is assumed in the given quotation.

As was shown in Section 2, all the conditions of equilibrium are completely satisfied in the classical theory of plates.

We will give another example which refutes the above hypothesis. The equations of the theory of elasticity in displacements are of the sixth order. By the logic used in [2] they should be of the twelfth order: three equations of equilibrium in projections onto the x, y and z axes plus three equations of the balance of the moments about these axes, and all these multiplied by two.

Generally speaking, the idea itself of finding a certain principle which rigorously relates the physical models (a continuous medium, equilibrium, etc.) to the mathematical models (the type and order of the equations, etc.) is doubtful. However, the same physical model—a continuous elastic plate, can have different mathematical models —a membrane (a second-order equation), the classical theory of plates (a fourth-order equation), and the Reissner theory of plates (a sixth-order equation).

An illustration that the condition of the balance of shear forces is not satisfied, the problem of the torsion of a plate by forces at the corners was considered in [2] (see above). "We will now consider a quarter of the plate $0 \leq x \leq a/2$, $0 \leq y \leq b/2$. We obtain a plate on which only a force $2Q$, concentrated at the corner, acts. No other transverse forces act on it. The balance of the shear forces is obviously violated" [2]. The author, however, overlooks the fact that when the quarter is cut, three new angles are produced at which the same concentrated forces act as on the initial plate.

The problem of a beam. The essence of the contradiction pointed out [2] is the fact that for problems of the bending of beams the same deflections, but different shear forces are obtained by statically equivalent loads. In Fig. 5 we show diagrams of Q and M for this case. In the traditional description of the classical theory of plates the shear forces Q and the bending moments M, when there is a distributed moment load m, are related by the formula

$$Q = \partial M / \partial x + m \tag{3.3}$$

and the contradiction is obvious.

Equation (3.3) is obtained from a consideration of the equilibrium of an element of the beam on which the distributed moment load acts. Then, since $Q \overset{\text{def}}{=} \int \tau_{xz} dz$ we can assume that Q and M act simultaneously. If we dispense with the assumption in Section 2 of interpreting the shear force as a "turning" moment, we have instead of (3.3)

$$Q \overset{\text{def}}{=} \partial M / \partial x \tag{3.4}$$

and there is no contradiction for either version of the loading. From the point of view of this treatment of the classical theory of plates the difference in the diagrams in Fig. 5 is due to the fact that in one case (on the right in Fig. 5) when drawing the diagram of Q the moments were "rotated", while in the second they were not. Generally speaking, it is advisable to rotate the external distributed moments immediately and to consider the generalized shear load as in (2.3).

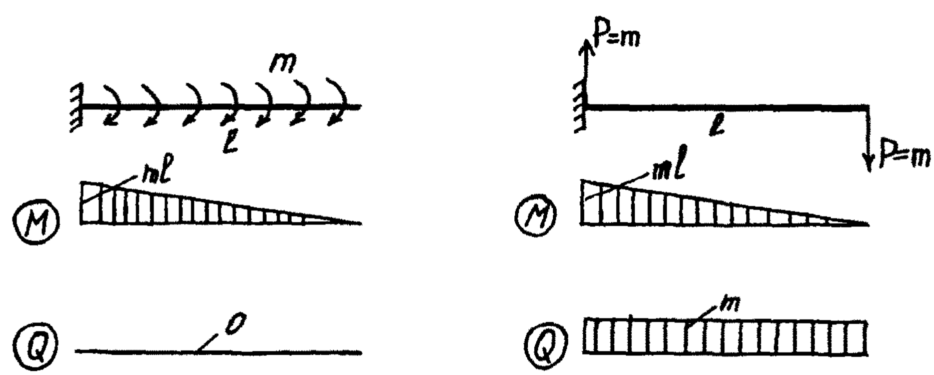

Fig. 5.

The problems of a punch. "An example of the solution which does not agree with the physical meaning of the problem is the problem of a plate bent by a shallow punch, the surface of which is given by the equation $z = a_0 + a_1x^2 + a_2y^2$. It is obvious that in the law of contact with a punch the bending deflection will also be determined by a second-order polynomial". Hence it follows that the contact pressure found from the Germain–Lagrange equation is not present, and "the equilibrium of the punch is ensured by forces distributed over the boundary of the region of contact" [1].

In the above discussions it was assumed that it is obviously possible for a region of contact to exist between the punch and the plate. However, this assumption is of doubtful validity.

The loading of a circular plate by an (for simplicity) axisymmetric punch begins with the application of a concentrated force at the centre of the plate at the point of initial contact with the punch. The deflection function in this case has the form [4]: $w_1 = b_0 + b_1r^2 + b_2r^2\ln r^2$, where $r^2 = x^2 + y^2$ and b_0, b_1 and b_2 are constants. This expression is not a second-order polynomial and there is no region of contact. When the pressure on the punch is increased a circle of contact occurs, and the contact forces are now transmitted over the whole of this circle. The depressions from the side external to the circle are then deflections by an expression of the type w_1, while inside the circle it is determined by the expression $w_0 = c_0 + c_1r^2$, which under the conditions $c_0 = a_0$, $c_1 = a_1 = a_2$ (for a certain value of the pressure of the punch) coincides with the punch surface. But when the load is increased further this coincidence breaks down and new circles of contact occur, etc. Hence, the calculation procedure changes during the loading. It was suggested in [7] that this kind of problem should be called a structurally non-linear problem.

If, in the case considered, it is assumed that there is a local region of contact then it is possible to analyse the cylindrical bending using two punches [3] (Fig. 6a). Here again we have a structurally non-linear problem. At the initial instant, due to the action of the concentrated force at the point A, a freely supported plate is bent in the form of a cubic parabola (Fig. 6b). For punches in the form of a quadratic parabola, when P reaches a certain value, new points of contact arise, etc. Once again the theoretical scheme changes during the loading. When $P = P^*$ the system acquires the form shown in Fig. 6(a).

The following contradictions were pointed out in [3]. It can be seen from Fig. 6(a) and (b) that a freely supported plate is bent by the normal pressure of the punch. But, on substituting an expression for the deflections of the form $w_3 = d_0 + d_1x^2$ into the boundary-value problem of the classical theory of plates, one can obtain that for such a bending of the plate additional contour moments are necessary which do not exist in the case of free support.

In these discussions the fact that, at the end of the loading, the plate is no longer freely supported, is overlooked. It is clamped. It is precisely this clamping which produces the bending moment which enables the plate to take the form of a quadratic parabola. It can be seen from Fig. 6(c) that if we "remove" the material of the punches everywhere apart from the boundary points, we will in no way affect the elastic line of the plate, since it is completely defined by the moments produced by the clamp. Hence there is no contradiction.

Determination of the reactions of the supports. When determining the reactions of the supports of a rectangular plate, hinge-supported along the contour, and loaded with a uniform pressure, concentrated forces occur at the corner points which press the plate to the contour. However, "the presence of such forces is not confirmed by the solution of the three-dimensional problem . . . , i.e. the solution obtained using the classical theory of plates does not agree with the theory of elasticity" [1]. In the theory of elasticity a free support is modelled by means of the conditions on the contour x = const

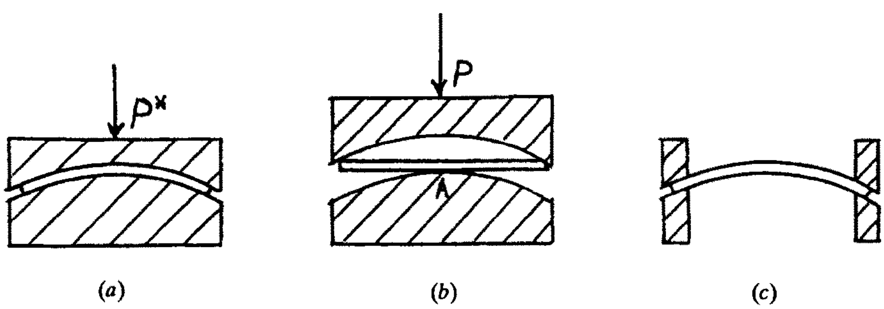

Fig. 6.

$$\sigma_x = 0,\ u_z = 0,\ u_y = 0 \tag{3.5}$$

Whereas the first two conditions do not give rise to any doubts, since they correspond in the classical theory of plates to the conditions

$$M_x = 0,\ w = 0 \tag{3.6}$$

the last condition in (3.5) is not quit so obvious. As was pointed out in Section 1, the absence of shear displacements in the median plane of the plate for bending problems is due to the fact that these displacements are not zero, and the fact that in order to determine them one must solve a separate plane problem. In the case of a free support for the plane problem along the contour it is logical to specify the following conditions

$$N_x = \int \sigma_x dz = 0,\ N_{xy} = \int \tau_{xy} dz = 0 \tag{3.7}$$

The first condition of (3.7) is also taken into account in (3.5) by the first condition, while the second condition in (3.7) in the solution of the three-dimensional problem must be taken into account by replacing the condition $u_y = 0$ in (3.5) by

$$\tau_{xy} = 0 \tag{3.8}$$

But, as was pointed out in [1], the first two conditions of (3.5) and condition (3.8) along the contour x = const when solving the three-dimensional problem lead to the occurrence of attractive forces.

Generally speaking, irrespective of how one models a free support in the theory of elasticity, the determination of the reactions is fundamentally inaccurate. This is due to the fact that in regions of local application of the load the stress–strain state changes rapidly in all directions and hence the classical theory of plates is inapplicable. The same also applies to more advanced theories of plates as a consequence of the approximate nature of the theory, and not its internal contradictions. It may be some consolation in engineering calculations that the amplitudes of the reactions of the supports, determined using the classical theory of plates, are higher than actually occurs (see [3]), and hence the errors are contained in the safety factor.

REFERENCES

1. VASIL'YEV V. V., On the theory of thin plates. *Izv. Ross. Akad. Nauk MTT* 3, 26–47, 1992.
2. ZHILIN P. A., The Poisson and Kirchhoff theories of thin plates from the point of view of the modern theory of plates. *Izv. Ross. Akad. Nauk. MTT* 3, 48–64, 1992.
3. ALFUTOV N. A., Some paradoxes of the theory of thin elastic plates. *Izv. Ross. Akad. Nauk. MTT* 3, 65–72, 1992.
4. TIMOSHENKO S. P. and WOINOWSKY-KRIEGER S., *Plates and Shells*. Fizmatgiz, Moscow, 1963.
5. VASIDZU K., *Variational Methods in the Theory of Elasticity and Plasticity*. Mir, Moscow, 1987.
6. PANOVKO Ya. G., *Mechanics of a Deformed Solid*. Nauka, Moscow, 1985.
7. LUKASH P. A., *Principles of Non-linear Structural Mechanics*. Stroiizdat, Moscow, 1978.

Translated by R.C.G.